Int. J. Heat Mass Transfer. Vol. 16, pp. 555–565. Pergamon Press 1973. Printed in Great Britain

HEAT TRANSFER THROUGH THE UNSTEADY LAMINAR BOUNDARY LAYER ON A SEMI-INFINITE FLAT PLATE PART I: THEORETICAL CONSIDERATIONS

NORMAN A. EVANS*

Gas Dynamics Laboratory, Forrestal Campus, Princeton, N.J., U.S.A.

(*Received* 28 *February* 1972 *and in revised form* 17 *July* 1972)

Abstract—The problem of a semi-infinite plate moving with a velocity $u_w(t)$ into a stagnant fluid is considered, and a solution correct to second order for heat transfer with constant surface enthalpy is obtained for small $\xi = x/\int_0^t u_e(t)\,dt$ (where t is time, $u_e(t) = -u_w(t)$ and x is the distance from the leading edge). The solution, which includes the effect of viscous dissipation, is valid under the conditions that $u_e(t) > 0$ and $u_e(t)$ is infinitely differentiable for all t, and is the continuation of the work of Cheng and Elliott, who obtained the solution to the incompressible momentum equation for the velocity profile and skin friction. Second order results were obtained by numerical integration for a Prandtl number of 0·72, and satisfactory agreement is shown with the results of other investigators who used different approaches in obtaining just the first order solution, or an incomplete second order solution.

NOMENCLATURE

a,	speed of sound;
c,	constant in viscosity–enthalpy law;
f, F,	functions in expansion of ψ;
h,	enthalpy;
H,	$(h - h_\infty)/(h_w - h_\infty)$;
M,	Mach number;
p,	pressure;
Pr,	Prandtl number;
q,	heat transfer rate;
r, R, λ,	functions in expansion of H;
S,	$\omega x/u_\infty$;
t,	time;
$u_e(t)$,	external fluid velocity;
u,	velocity in x direction;
v,	velocity in y direction;
w, W, χ,	functions in expansion of H;
x,	distance from leading edge of plate;
y,	distance normal to plate surface;
α_0^2,	$u_e' \int_0^t u_e\,dt/u_e^2$;
ϵ,	constant;
μ,	viscosity.

Greek symbols

ν,	kinematic viscosity;
ψ,	stream function;
ρ,	density;
ξ,	$x/\int_0^t u_e\,dt$;
ξ_0,	$\xi^{\frac{1}{2}}$;
ζ_0,	$y(u_e/c\nu_\infty x)^{\frac{1}{2}}$;
ω,	circular frequency.

Subscripts

w,	plate surface;
∞,	undisturbed fluid;
K,	summation index;
s,	steady.

Superscripts

*,	refers to coordinate axes fixed in space.

* Present address: Towne School of Civil and Mechanical Engineering, University of Pennsylvania, Philadelphia, Pa.

1. INTRODUCTION AND METHOD OF APPROACH

VARIOUS investigators have analyzed unsteady flat plate motions, the earliest being Stokes [1], followed by Rayleigh [1], with their associated classical problems which yielded exact solutions to the incompressible Navier–Stokes equations. Impulsively started compressible flat plate motions have been studied by Stewartson [2, 3], Howarth [4], Van Dyke [5] and others, and Ostrach [6] has investigated the compressible analogue to Stokes' problem.

Unsteady solutions for skin friction for various types of semi-finite plate motion, $u_w(t)$, have been obtained by Lighthill [7], Moore [8], Cheng [9] and Cheng and Elliott [10]. Sarma, in a series of papers [11–14], developed a unified theory for unsteady boundary layers under varying conditions of velocity and temperature based on a linearized analysis similar to that of Lighthill [7]. The present paper is an extension of Cheng and Elliott's work to include compressible heat transfer effects. The expansion solution obtained for constant plate surface enthalpy is valid for small $\xi = x/t_0^t u_e(t)\,dt$ (where t is time, $u_e(t) = -u_w(t)$ and x is the distance from the leading edge), under the conditions that $u_e(t) > 0$ and $u_e(t)$ is infinitely differentiable for all t. These conditions stipulate that the plate moves with no reverse flow (i.e. the leading edge never becomes a trailing edge), and that motions of an impulsive nature are excluded. Cheng and Elliott's further condition that the plate start from rest ($u_e(0) = 0$) will be relaxed, so that the heat transfer will consist of the quasi-steady value followed by terms representing deviations from this value.

After reduction of the appropriate compressible boundary layer equations to an incompressible, uncoupled form by means of the Howarth–Dorodnitsyn variable under the assumption of a linear viscosity–enthalpy law, two series expansions in $\xi_0 = \xi^{\frac{1}{2}}$ are developed for the energy equation. The first series gives the heat transfer rate occurring under a constant driving potential in the absence of viscous dissipation, the correction due to this effect being made by addition of the results from the second series. The validity of this procedure follows from the linear nature of the energy equation.

The results obtained are then compared with those of Lighthill [7], Ostrach [15] and Illingworth [16], the nature of whose work will be described when the comparison is made in a later section.

Part II will present experimental results for verification of the theory.

2. FORMULATION OF THE PROBLEM

For a flat plate with zero pressure gradient and constant surface enthalpy, moving unsteadily into a compressible stagnant fluid, where the coordinate axes (x^*, y^*) are fixed in space, the boundary layer equations are:

Continuity:

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial x^*}(\rho^* u^*) + \frac{\partial}{\partial y}(\rho^* v^*) = 0 \qquad (2.1)$$

Momentum:

$$\rho^* \frac{\partial u^*}{\partial t^*} + \rho^* u^* \frac{\partial u^*}{\partial x^*} + \rho^* v^* \frac{\partial u^*}{\partial y^*} = \frac{\partial}{\partial y^*}\left(\mu^* \frac{\partial u^*}{\partial y^*}\right) \qquad (2.2)$$

$$\frac{\partial p^*}{\partial y^*} = 0$$

Energy:

$$\rho^* \frac{\partial h^*}{\partial t^*} + \rho^* u^* \frac{\partial h^*}{\partial x^*} + \rho^* v^* \frac{\partial h^*}{\partial y^*} = \frac{1}{Pr}\frac{\partial}{\partial y^*}\left(\mu^* \frac{\partial h^*}{\partial y^*}\right) + \mu^* \left(\frac{\partial u^*}{\partial y^*}\right)^2 \qquad (2.3)$$

where (u^*, v^*) are the velocity components in the (x^*, y^*) directions respectively, and ρ^*, μ^*, p^* and h^* are respectively the density, viscosity, pressure and enthalpy, and Pr is the Prandtl number.

To solve the above system, the following equations are added:

State: $$p^* = \rho^* R T^* \tag{2.4}$$

Viscosity–enthalpy law:

$$\mu^* = \mu^*(h^*). \tag{2.5}$$

For all t^* the plate is considered to be lying along the x^*-axis ($y^* = 0$) with its leading edge at $x^* = 0$ when $t^* = 0$, and, at this instant, the plate is moving with a velocity $u_w^*(t^*)$ in the negative x^*-direction. This velocity is arbitrary to the extent allowed by the following conditions:

(1) $u_w^*(t^*) < 0$ for all t^*

(2) $u_w^*(t^*)$ is infinitely differentiable for all t^*.

Condition (1) ensures that the plate moves with no ensuing reverse motion, and condition (2) excludes motions of an impulsive nature.

Following the development of Cheng and Elliott [10], use of the Howarth–Dorodnitsyn variable, introduction of a stream function defined by

$$\frac{\rho^*}{\rho^*_\infty} u^* = \frac{\partial \psi^*}{\partial y^*},$$

use of a linear viscosity enthalpy relation of the form

$$\frac{\mu^*}{\mu^*_\infty} = c\frac{h^*}{h^*_\infty}$$

(c is a constant), and transfer of the coordinate system (x^*, y^*, t^*) fixed in space to (x, y, t) fixed in the plate with the origin at the leading edge and $u_w^*(t^*) = -u_e(t)$, transforms the equation system (2.1)–(2.5) to

$$\psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} = \frac{du_e}{dt} + cv_\infty\psi_{yyy} \tag{2.6}$$

$$h_t + \psi_y h_x - \psi_x h_y = \frac{cv_\infty}{Pr} h_{yy} + cv_\infty \psi_{yy}^2. \tag{2.7}$$

The appropriate boundary conditions are

$$\psi_x = \psi_y = 0 \quad \text{for} \quad x \geqslant 0, y = 0 \tag{2.8}$$

$$h = h_w = \text{constant for } x \geqslant 0, y = 0 \tag{2.9}$$

$$\left.\begin{aligned}\psi_y &= u_e(t)\\ h &= h_\infty\end{aligned}\right\} \quad \text{for} \quad x \geqslant 0, y \to \infty. \tag{2.10}$$

Equations (2.6) and (2.7) are uncoupled and in incompressible form, and solutions obtained for (2.7) are valid both when the plate is moving into undisturbed fluid and when the fluid is moving over the plate at rest. However, any solution obtained, and applied (via the inverse transformation) to the full compressible problem, is valid only for the plate moving unsteadily into fluid at rest; this can be seen from the boundary condition (2.10), since h_∞ can be constant only in a stagnant (compressible) field.

3. NON-DIMENSIONAL PARAMETERS FOR THE SOLUTION FOR SMALL ξ

From the work of Cheng [9] and Cheng and Elliott [10], it turns out that the correct parameters for a solution around $\xi = 0$ are:

$$\xi_0 = \xi^{\frac{1}{2}} = \left(\frac{x}{\int_0^t u_e(t)\,dt}\right)^{\frac{1}{2}}$$

$$\text{and} \quad \zeta_0 = \eta/\alpha_0\xi_0 = y\left(\frac{u_e}{cv_\infty x}\right)^{\frac{1}{2}} \tag{3.1}$$

where

$$\alpha_0^2 = \frac{u^e(t)\int_0^t u_e(t)\,dt}{u_e^2} \tag{3.2}$$

with the stream function expanded as

$$\psi = (cv_\infty u_e x)^{\frac{1}{2}} \sum_{K=0}^{\infty} F_K(\zeta_0, t)\xi_0^K. \tag{3.3}$$

As defined in equation (3.1), ζ_0 is a direct extension of the Blasius parameter into unsteady flow. The parameter α_0^2, equation (3.2), is required in the ensuing analysis.

Following a procedure similar to that used by Ostrach [11], a non-dimensional parameter for the energy equation, with constant surface enthalpy, h_w, is defined by:

$$H(x, y, t) = \frac{h(x, y, t) - h_\infty}{h_w - h_\infty}. \tag{3.4}$$

Substitution of (3.3) and (3.4) into (2.13) yields:

$$H_t + \psi_y H_x - \psi_x H_y = \frac{cv_\infty}{Pr} H_{yy} + \frac{cv_\infty}{h_w - h_\infty}\psi_{yy}^2. \tag{3.5}$$

Equation (3.5) is the final form of the energy equation, which is linear in H. For constant surface enthalpy a solution will first be obtained to the homogeneous equation involving the storage, convection and conduction terms, and to this will be added the particular integral from the full inhomogeneous equation which will include the viscous dissipation term. Hence H is split as follows:

$$H(x, y, t) = w(x, y, t) + \frac{u_e^2}{2(h_w - h_\infty)} r(x, y, t) \quad (3.6)$$

where w is the homogeneous solution and r is the particular integral. Finally, w and r are expanded as:

$$w = \sum_{K=0}^{\infty} W_K(\zeta_0, t)\, \xi_0^K \quad (3.7)$$

$$r = \sum_{K=0}^{\infty} R_K(\zeta_0, t)\, \xi_0^K. \quad (3.8)$$

4. DEVELOPMENT OF THE EQUATIONS FOR SMALL ξ

From the parameters defined in (3.1), the transformation relations from (x, y, t) to (ξ_0, ζ_0, t) are:

$$\frac{\partial}{\partial x} = \frac{1}{2}\frac{\xi_0}{x}\frac{\partial}{\partial \xi_0} - \frac{1}{2}\frac{\zeta_0}{x}\frac{\partial}{\partial \zeta_0}; \quad \frac{\partial}{\partial y} = \left(\frac{u_e}{cv_\infty x}\right)^{\frac{1}{2}} \frac{\partial}{\partial \zeta_0};$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \frac{u_e \xi_0}{2}\frac{1}{\int_0^t u_e\, dt}\frac{\partial}{\partial \xi_0} + \frac{1}{2}\frac{u_e^1}{u_e}\zeta_0\frac{\partial}{\partial \zeta_0}. \quad (4.1)$$

Introduction of ψ from (3.3), w from (3.7) and r from (3.8) into the momentum equation (2.12) and the energy equation (3.5) (neglecting the viscous dissipation term when substituting for w), results in the following systems of partial differential equations:

$$\sum_{K=0}^{\infty}\left\{\frac{\partial}{\partial t}\left(u_e \frac{\partial F_K}{\partial \zeta_0}\right) - \frac{1}{2}\frac{u_e^2}{\int_0^t u_e\, dt} K \frac{\partial F_K}{\partial \zeta_0} + \frac{u_e'}{2}\zeta_0 \frac{\partial^2 F_K}{\partial \zeta_0^2}\right\}\xi_0^K + \frac{u_e^2}{2\int_0^t u_e\, dt}$$

$$\times \left\{\left(\sum_{K=0}^{\infty}\frac{\partial F_K}{\partial \zeta_0}\xi_0^K\right)\left(\sum_{K=0}^{\infty} K \frac{\partial F_K}{\partial \zeta_0}\xi_0^{K-2}\right) - \left(\sum_{K=0}^{\infty}\frac{\partial^2 F_K}{\partial \zeta_0^2}\xi_0^{K-1}\right)\left(\sum_{K=0}^{\infty}(K+1) F_K \xi_0^{K-1}\right)\right\}$$

$$= u_e' + \frac{u_e^2}{\int_0^t u_e\, dt}\sum_{K=0}^{\infty}\frac{\partial^3 F_K}{\partial \zeta_0^3}\xi_0^{K-2} \quad (4.2)$$

$$\frac{\int_0^t u_e\, dt}{u_e}\sum_{K=0}^{\infty}\left\{\frac{\partial W_K}{\partial t} - \frac{1}{2}\frac{u_e}{\int_0^t y_e\, dt} K W_K + \frac{1}{2}\frac{u_e'}{u_e}\zeta_0\frac{\partial W_K}{\partial \zeta_0}\right\}\xi_0^K + \frac{1}{2}\left\{\left(\sum_{K=0}^{\infty}\frac{\partial F_K}{\partial \zeta_0}\xi_0^K\right)\right.$$

$$\left.\times\left(\sum_{K=0}^{\infty} K W_K \xi_0^K\right) - \left[\sum_{K=0}^{\infty}(K+1) F_K \xi_0^K\right]\right.$$

$$\left.\times\left(\sum_{K=0}^{\infty}\frac{\partial W_K}{\partial \zeta_0}\xi_0^K\right)\right] = \frac{1}{Pr}\sum_{K=0}^{\infty}\frac{\partial^2 W_K}{\partial \zeta_0^2}\xi_0^K \quad (4.3)$$

$$\frac{\int_0^t u_e dt}{u_e}\sum_{K=0}^{\infty}\left\{\frac{\partial R_K}{\partial \zeta_0} - \frac{1}{2}\frac{u_e}{\int_0^t u_e\, dt} K R_K + \frac{1}{2}\frac{u_e'}{u_e}\zeta_0\frac{\partial R_K}{\partial \zeta_0} + 2\frac{u_e'}{u_e} R_K\right\}\xi_0^{K+2}$$

$$+\frac{1}{2}\left\{\left(\sum_{K=0}^{\infty}\frac{\partial F_K}{\partial \zeta_0}\xi_0^K\right)\left(\sum_{K=0}^{\infty} K R_K \xi_0^K\right) - \left[\sum_{K=0}^{\infty}(K+1) F_K \xi_0^K\right]\left(\sum_{K=0}^{\infty}\frac{\partial R_K}{\partial \zeta_0}\xi_0^K\right)\right\}$$

$$= \frac{1}{Pr}\sum_{K=0}^{\infty}\frac{\partial^2 R_K}{\partial \zeta_0^2}\xi_0^K + 2\left(\sum_{K=0}^{\infty}\frac{\partial^2 F_K}{\partial \zeta_0^2}\xi_0^K\right)^2. \quad (4.4)$$

By the coordinate transformation of (4.1),

$$\frac{u}{u_e} = \sum_{K=0}^{\infty}\frac{\partial F_K}{\partial \zeta_0}\xi_0^K \quad \text{and}$$

$$v = \frac{1}{2}\left(\frac{cv_\infty u_e}{x}\right)^{\frac{1}{2}} \times \sum_{K=0}^{\infty}\left[\zeta_0\frac{\partial F_K}{\partial \zeta_0} - (K+1) F_K\right]\xi_0^K.$$

Hence, to satisfy boundary condition (2.8), it will be necessary that

$$F_K(0, t) = 0 = \frac{\partial F_K}{\partial \zeta_0}(0, t).$$

From (3.4), (3.6), (3.7) and (3.8) with $h(x, 0, t) = h_w$ = constant, then

$$H(x, 0, t) = 1 = \sum_{K=0}^{\infty} W_K(0, t)\,\xi_0^K + \frac{u_e^2}{2(h_w - h_\infty)} \times \sum_{K=0}^{\infty} R_K(0, t)\,\xi_0^K.$$

Since H at the plate surface is constant, and therefore independent of and u_e, it follows that

$$W_K(0, t) = \delta_{0K} \quad \text{and}$$

$$R_K(0, t) = 0 \quad \text{to satisfy (2.9)}$$

Finally, to satisfy (2.10) at the edge of the boundary layer,

$$\frac{\partial F_K}{\partial \zeta_0}(\infty, t) = \delta_{0K}$$

$$\text{and} \qquad W_K(\infty, t) = 0 = R_K(\infty, t).$$

In each of the equations (4.2), (4.3) and (4.4), the coefficient of each power of ξ_0 is equated to zero, and this produces three sets of linear partial differential equations (except that for F_0 which is non-linear).

Equation (4.2) for the F_K's was solved by Cheng and Elliott [10] and will not be further discussed here, although it will, of course, be necessary to use their results for the solution of equations (4.3) and (4.4).

The solution to the energy equation is now developed. Primes denote differentiation with respect to ζ_0, or, when the argument is t only, with respect to t.

Zeroth order equations

$$\frac{1}{Pr} W_0'' + \tfrac{1}{2} F_0 W_0' = 0$$

with

$$W_0(0, t) = 1, \qquad W_0(\infty, t) = 0 \tag{4.5}$$

and

$$\frac{1}{Pr} R_0'' + \frac{1}{2} F_0 R_0' = -2(F_0'')^2,$$

with

$$R_0(0, t) = 0, \qquad R_0(\infty, t) = 0 \tag{4.6}$$

where [10]

$$F_0(\zeta_0, t) = f_0(\zeta_0) \quad \text{only.}$$

The nature of (4.5) and (4.6) with their boundary conditions indicate the W_0 and R_0 are functions of ζ_0 only, i.e.

$$W_0(\zeta_0, t) = \lambda_0(\zeta_0) \quad \text{and} \quad R_0(\zeta_0, t) = \chi_0(\zeta_0).$$

Hence (4.5) and (4.6) become

$$\frac{1}{Pr} \lambda_0'' + \tfrac{1}{2} f_0 \lambda_0' = 0$$

with

$$\lambda_0(0) = 1, \qquad \lambda_0(\infty) = 0, \tag{4.7}$$

and

$$\frac{1}{Pr} \chi_0'' + \tfrac{1}{2} f_0 \chi_0' = -2(f_0'')^2$$

with

$$\chi_0(0) = 0, \qquad \chi_0(\infty) = 0. \tag{4.8}$$

First order equations

Since [10]

$$F_1(\zeta_0, t) = 0,$$

then

$$\frac{1}{Pr} W_1'' + \tfrac{1}{2} F_0 W_1' - \tfrac{1}{2} F_0' W_1 = 0$$

with

$$W_1(0, t) = 0 = W_1(\infty, t), \tag{4.9}$$

and

$$\frac{1}{Pr} R_1'' + \tfrac{1}{2} F_0 R_1' - \tfrac{1}{2} F_0' R_1 = 0$$

with

$$R_1(0, t) = 0 = R_1(\infty, t). \tag{4.10}$$

Since equations (4.9) and (4.10) are homogeneous with homogeneous conditions, it appears that the solutions are given by $W_1(\zeta_0, t) = 0$ and $R_1(\zeta_0, t) = 0$. For all odd values of K, the W_K and R_K equations are given by (using the fact that, for odd K, $F_K(\zeta_0, t) = 0$):

$$\frac{1}{Pr} W''_K + \tfrac{1}{2} F_0 W'_K - \frac{K}{2} F'_0 W_K = 0$$

with

$$W_K(0, t) = 0 = W_K(\infty, t)$$

and

$$\frac{1}{Pr} R''_K + \tfrac{1}{2} F_0 R'_K - \frac{K}{2} F'_0 R_K$$

with

$$R_K(0, t) = 0 = R_K(\infty, t).$$

The solutions are given by $W_K(\zeta_0, t) = 0$ and and $R_K(\zeta_0, t) = 0$ for K odd.

Second order equations

$$\frac{1}{Pr} W''_2 + \tfrac{1}{2} F_0 W'_2 - F'_0 W_2 = \tfrac{1}{2}\alpha_0^2 \zeta_0 W'_0 - \tfrac{3}{2} W'_0 F_2$$

with

$$W_2(0, t) = 0 = W_2(\infty, t)$$

and

$$\frac{1}{Pr} R''_2 + \tfrac{1}{2} F_0 R'_2 - F'_0 R_2 = \tfrac{1}{2}\alpha_0^2 \zeta_0 R'_0 + 2\alpha_2^2 R_0 - \tfrac{3}{2} R'_0 F_2$$

with

$$R_2(0, t) = 0 = R_2(\infty, t).$$

Since [10] $F_2(\zeta_0, t) = \alpha_0^2(t) f_2(\zeta_0)$, and writing $W_2(\zeta_0, t) = \alpha_0^2(t)\lambda_2(\zeta_0)$ the equation to be satisfied by λ_2 is:

$$\frac{1}{Pr}\lambda''_2 + \tfrac{1}{2} f_0 \lambda'_2 - f'_0 \lambda_2 = \tfrac{1}{2}(\zeta_0 - 3f_2)\lambda'_0 \quad (4.11)$$

with

$$\lambda_2(0) = 0 = \lambda_2(\infty).$$

Similarly, writing $R_2(\zeta_0, t) = \alpha_0^2(t)\chi_2(\zeta_0)$, the equation to be satisfied by χ_2 is:

$$\frac{1}{Pr}\chi''_2 + \tfrac{1}{2} f_0 \chi'_2 - f'_0 \chi_2 = \tfrac{1}{2}(\zeta_0 - 3f_2)\chi'_0 + 2\chi_0 - 4f''_0 f''_2 \quad (4.12)$$

with

$$\chi_2(0) = 0 = \chi_2(\infty).$$

Fourth order equations

$$\frac{1}{Pr} W''_4 + \tfrac{1}{2} F_0 W'_4 - 2F'_0 W_4 = (F'_2 - 1) W_2 + \tfrac{1}{2}(\alpha_0^2\zeta_0 - 3F_2) W'_2 + \frac{\int_0^t u_e\,dt}{u_e}\frac{\partial W_2}{\partial t} - \tfrac{5}{2} F_4 W'_0$$

with

$$W_4(0, t) = 0 = W_4(\infty, t)$$

and

$$\frac{1}{Pr} R''_4 + \tfrac{1}{2} F_0 R'_4 - 2F'_0 R_4 = (F'_2 - 1 + 2\alpha_0^2) R_2 + \tfrac{1}{2}(\alpha_0^2\zeta_0 - 3F_2) R'_2 + \frac{\int_0^t u_e\,dt}{u_e}\frac{\partial R_2}{\partial t} - \tfrac{5}{2} F_4 R'_0 - 2(F''_2)^2 - 4F''_0 F''_4$$

with

$$R_4(0, t) = 0 = R_4(\infty, t).$$

Since [10]

$$F_4(\zeta_0, t) = \frac{u''_e}{u_e^3}\left[\int_0^t u_e\,dt\right]^2 f_{41}(\zeta_0) + \frac{(u'_e)^2}{u_e^4}\left[\int_0^t u_e\,dt\right]^2 f_{42}(\zeta_0),$$

and writing

$$W_4(\zeta_0, t) = \alpha_0^4(t)\lambda_{41}(\zeta_0) + \left[\frac{\int_0^t u_e\,dt}{u_e}\frac{d\alpha_0^2}{dt} - \alpha_0^2\right] \times \lambda_{42}(\zeta_0) - \frac{u''_e}{u_e^3}\left[\int_0^t u_e\,dt\right]^2 \lambda_{43}(\zeta_0) - \frac{(u'_e)^2}{u_e^4}\left[\int_0^t u_e\,dt\right]^2 \lambda_{44}(\zeta_0)$$

the equations to be satisfied by λ_{41}, λ_{42}, λ_{43} and λ_{44} are:

$$\frac{1}{Pr}\lambda''_{41} + \tfrac{1}{2}f_0\lambda'_{41} - 2f'_0\lambda_{41} = \tfrac{1}{2}(\zeta_0 - 3f_2)\lambda'_2 + f'_2\lambda_2 \quad (4.13)$$

$$\frac{1}{Pr}\lambda''_{42} + \tfrac{1}{2}f_0\lambda'_{42} - 2f'_0\lambda_{42} = \lambda_2 \quad (4.14)$$

$$\frac{1}{Pr}\lambda''_{43} + \tfrac{1}{2}f_0\lambda'_{43} - 2f'_0\lambda_{43} = \tfrac{5}{2}f_{41}\lambda'_0 \quad (4.15)$$

$$\frac{1}{Pr}\lambda''_{44} + \tfrac{1}{2}f_0\lambda'_{44} - 2f'_0\lambda_{44} = \tfrac{5}{2}f_{42}\lambda'_0 \quad (4.16)$$

with

$$\lambda_{4\phi}(0) = 0 = \lambda_{4\phi}(\infty) \qquad (\phi = 1, 2, 3 \text{ or } 4).$$

Similarly, writing

$$R_4(\zeta_0, t) = \alpha_0^4(t)\,\chi_{41}(\zeta_0) + \left(\frac{\int_0^t u_e\,dt}{u_e}\frac{d\alpha_0^2}{dt} - \alpha_0^2\right)\chi_{42}(\zeta_0) - \frac{u''_e}{u_e^3}\Big[\int_0^t u_e\,dt\Big]^2\chi_{43}(\zeta_0) - \frac{(u'_e)^2}{u_e^4}\Big[\int_0^t u_e\,dt\Big]^2\chi_{44}(\zeta_0)$$

the equations to be satisfied by χ_{41}, χ_{42}, χ_{43}, χ_{44} are

$$\frac{1}{Pr}\chi''_{41} + \tfrac{1}{2}f_0\chi'_{41} - 2f'_0\chi_{41} = \tfrac{1}{2}(\zeta_0 - 3f_2)\chi_2 + (f'_2 + 2)\chi_2 - 2(f''_2)^2 \quad (4.17)$$

$$\frac{1}{Pr}\chi''_{42} + \tfrac{1}{2}f_0\chi'_{42} - 2f'_0\chi_{42} = \chi_2 \quad (4.18)$$

$$\frac{1}{Pr}\chi''_{43} + \tfrac{1}{2}f_0\chi'_{43} - 2f'_0\chi_{43} = \tfrac{5}{2}f_{41}\chi'_0 + 4f''_0 f''_{41} \quad (4.19)$$

$$\frac{1}{Pr}\chi''_{44} + \tfrac{1}{2}f_0\chi'_{44} - 2f'_0\chi_{44} = \tfrac{5}{2}f_{42}\chi'_0 + 4f''_0 f''_{42} \quad (4.20)$$

with

$$\chi_{4\phi}(0) = 0 = \chi_{4\phi}(\infty) \qquad (\phi = 1, 2, 3 \text{ or } 4).$$

Numerical integration of equations (4.7), (4.8) and (4.11)–(4.20) was performed on an IBM 7090 computer with all equations, except (4.7), being integrated by the following method. For the solution, g, to the equation

$$D_K g = I \qquad \text{with} \quad g(0) = 0 = g(\infty)$$

where D_K is the linear second order operator

$$\frac{1}{Pr}\frac{d^2}{d\zeta_0^2} + \tfrac{1}{2}f_0\frac{d}{d\zeta_0} - \frac{K}{2}f'_0$$

and I is the inhomogeneous part with $I(\infty) = 0$, solutions are first found for

$$D_K g_0 = 0 \quad \text{with } g_0(0) = 0 \quad \text{and } g'_0(0) = 1$$

and

$$D_K\bar{g} = I \quad \text{with } \bar{g}(0) = 0 \quad \text{and } \bar{g}'(0) = A$$

where A is constant of suitable magnitude. Then the solution $g = \bar{g} - (\bar{g}(\infty)/g_0(\infty))\,g_0$ and the quantity associated with the surface heat transfer rate is

$$g'(0) = A - \frac{\bar{g}(\infty)}{g_0(\infty)}. \quad (4.21)$$

The actual integration scheme was of the Milne type [17], which automatically halved or doubled, when necessary, the stepwise interval in the independent variable in order to produce five significant figure accuracy. During the course of this part of the program, it was found possible to make slight improvements to Cheng and Elliott's [10] solutions for f_2, f_{41} and f_{42}.

Then, using the value obtained analogous to (4.21), the integration of the equations was repeated with a Runge–Kutta–Gill method, using an interval no larger than the smallest employed by the Milne method, in order to obtain machine output data at convenient intervals. On account of small differences produced by the two integration schemes used, minor adjustments were made to the values of $g'(0)$ so that $g(9.0)$ satisfied the values calculated independently from the asymptotic form of the governing equations. A check on the accuracy of this procedure was made by comparing the asymptotic value of the first derivative with that obtained from the machine computation. It was found that the values of $g'(0)$ were extremely sensitive in determining the asymptotic value of the function: in the fourth order equations, for instance, a change in $g'(0)$ of $\pm 1 \times 10^{-7}$ produced a change of $\pm 380 \times 10^{-7}$ in $g(9.0)$.

Due to space limitations, computer-printed tables of the results, to five decimal places, are available in [18] for $Pr = 0{\cdot}72$.

5. RESULTS AND DISCUSSION

After appropriate reduction, the expression for the surface heat transfer rate, for small ξ, is given by:

$$q(x, 0, t) = q_w = -\frac{c\mu_\infty}{Pr}\left(\frac{\partial h}{\partial y}\right)_{y=0}$$

$$= -\frac{1}{Pr}\left(\frac{u_e c\mu_\infty\rho_\infty}{x}\right)^{\frac{1}{2}}(h_w - h_\infty)\left(\frac{\partial H}{\partial \zeta_0}\right)_{\zeta_0=0} \tag{5.1}$$

Using the fact that the speed of sound, a_∞, in the undisturbed fluid is given by $a_\infty^2 = (\gamma - 1)\, h_\infty$, and defining an unsteady Mach number by $M_\infty(t) = u_e(t)/a_\infty$, equation (5.1) becomes, on substituting the numerical values found by integration for $Pr = 0{\cdot}72$:

$$q_w = 0{\cdot}41061\left(\frac{u_e c\mu_\infty\rho_\infty}{x}\right)^{\frac{1}{2}}(h_w - h_\infty)$$
$$\times \left\{1 - 0{\cdot}06906u_e'\frac{x}{u_e^2} + (0{\cdot}22312\,u_e'^2\right.$$
$$- 0{\cdot}42343u_e u_e'')\frac{x^2}{u_e^4} + \ldots - 0{\cdot}42386(\gamma - 1)M_\infty^2$$
$$\times \frac{h_\infty}{h_w - h_\infty}\left[1 + 0{\cdot}04587u_e'\frac{x}{u_e^2} + (1{\cdot}95014u_e'^2\right.$$
$$\left.\left. - 0{\cdot}54931u_e u_e'')\frac{x^2}{u_e^4} + \ldots\right]\right\}. \tag{5.2}$$

It is seen that the explicit dependence of q_w on ξ has dropped out. This independence of the implied time history (through the absence of $\int_0^t u_e\,dt$) was also found by Cheng and Elliott [10] in their solutions for velocity profile and skin friction, indicating that ξ is an indirect, rather than a direct, parameter for the unsteady boundary layer problem. However, it should be noted that the determination of q_w at a particular x and t immediately specifies ξ.

Further conditions for the validity of the present solution are:

(1) $u_e(t) > 0$ for all t
(2) $u_e(t)$ is infinitely differentiable for all t.

Ostrach [15] extended Moore's work [8] to include heat transfer effects by using a series of physically conceived parameters

$$\zeta_n = \frac{x^{n+1}u_e^{(n+1)}}{u_e^{n+2}}$$

in an expansion procedure. For comparison with equation (5.2), he obtained

$$q_w = 0{\cdot}4106\left(\frac{u_e c\mu_\infty\rho_\infty}{x}\right)^{\frac{1}{2}}(h_w - h_\infty)$$
$$\times \left\{1 - 0{\cdot}06923u_e'\frac{x}{u_e^2} - 0{\cdot}4232u_e''\frac{x^2}{u_e^3} + \ldots\right.$$
$$- 0{\cdot}4240(\gamma - 1)M_\infty^2\frac{h_\infty}{h_w - h_\infty}\left[1 - 0{\cdot}0448u_e'\right.$$
$$\left.\left.\times \frac{x}{u_e^2} - 0{\cdot}5493u_e''\frac{x^2}{u_e^3} + \ldots\right]\right\}. \tag{5.3}$$

In both (5.2) and (5.3), the leading terms of unity within the braces and square brackets represent, respectively, the quasi-steady heat transfer due to the driving potential $h_w - h_\infty$, and the viscous dissipation. Good agreement is found with those terms presented by Ostrach, the small differences presumably being due to errors in the integration schemes. Although Ostrach stated that he was interested only in the first order deviation (u_e^1/xu_e^2 terms) from the quasi-steady state, he included $u_e''x^2/u_e^3$ terms which are part of the second order solution based on the parameter x/u_e^2, as seen from the present solution (5.2). The complete second order solution requires inclusion of the terms $u_e'^2x^2/u_e^4$ in (5.2) which will, for example, give the only second order contribution for the case of constant acceleration ($u_e(t) = t$). The $u_e'^2x^2/u_e^4$ terms always give a positive contribution to q_w from the driving potential portion of the solution, and a similar situation was noted by Cheng and Elliot [10] when comparing their skin friction results with those of Moore [8]. The contribution from this term in the viscous dissipation portion of the solution is always negative when $h_w > h_\infty$.

Lighthill [7] and Illingworth [16] considered an unsteady velocity field of the form $u_e(t) = U_\infty(1 + \epsilon\, e^{i\omega t})$ where U_∞ is a steady stream velocity, ω is a circular frequency and ϵ is a non-dimensional constant with $\epsilon \ll 1$. Lighthill used a momentum integral technique, and Illingworth carried out an expansion in small $S = \omega x/U_\infty$. If q_{ws} is the heat transfer rate with the steady velocity U_∞ and dissipation is neglected, their results may be put in the form

$$\frac{q_w}{q_{ws}} = 1 + (0{\cdot}5 - BiS)\,\epsilon\, e^{i\omega t} \quad \text{for small } S \qquad (5.4)$$

where, by Lighthill, $B = 0{\cdot}03$ and, by Illingworth, $B = 0{\cdot}070$.

To the same order of accuracy (neglecting terms in S^2), the present analysis, and also that of Ostrach, give $B = 0{\cdot}069$. On account of the small magnitude of B, (5.4) shows that, for small S, the unsteady heat transfer rate and the velocity are essentially in phase.

6. CONCLUSIONS

The problem of heat transfer, including the effects of viscous dissipation, has been investigated for a semi-infinite flat plate moving unsteadily into a stagnant compressible fluid.

A series expansion solution, for small values of the parameter $\xi = x/\int_0^t u_e\, dt$ was constructed under the conditions that no flow reversal occurred and that the time-dependent plate velocity function was infinitely differentiable. The solution obtained consisted of the quasi-steady value followed by the first and second order unsteady deviations from it, both for the heat transfer occurring under a constant driving potential in the absence of viscous dissipation, and for the added correction due to this effect.

The final form of the solution was similar to that obtained by Ostrach for the same problem, using a different approach. The fact that the time-history dependent quantity ξ did not appear in the present solution indicates that it is an indirect parameter for the problem.

Good agreement was obtained with Ostrach's numerical values, but the present analysis contains a second order term not included by him. This term always gives a positive contribution to the driving potential portion of the solution and, for $h_w > h_\infty$, a negative contribution in the viscous dissipation portion of the solution.

To the first order, the present solution is also found to be in good agreement with the results of Lighthill and Illingworth who, by different, linearized, approaches, considered the case of a small amplitude sinusoidal velocity variation superimposed on a steady velocity.

ACKNOWLEDGEMENTS

This research was sponsored by the Fluid Mechanics Division of the Air Force Office of Scientific Research. The author wishes to express his gratitude to S. I. Cheng for discussion on the problem, and also to D. R. Willis and Mrs. Ruth V. Powers for their assistance in the programming for the numerical computations.

REFERENCES

1. S. Goldstein, *Modern Developments in Fluid Mechanics*. Clarendon Press (1938).
2. K. Stewartson, On the motion of a flat plate at high speed in a viscous compressible fluid, *Proc. Camb. Phil. Soc.* **51**, Part 1, 202–219 (1955).
3. K. Stewartson, On the impulsive motion of a flat plate in a viscous fluid, *Q. J. Mech. Appl. Math.* **4**, Part 2, 182–198 (June 1951).
4. L. Howarth, Some aspects of Rayleigh's problem for a compressible fluid, *Q. J. Mech. Appl. Math.* **4**, Part 2, 157–169 (June 1951).
5. M. D. Van Dyke, Impulsive motion of an infinite plate in a viscous compressible fluid, *Z. Angew. Math. Phys.* **III**, 343–353 (1952).
6. S. Ostrach, Note on the aerodynamic heating of an oscillating surface, NACA TN 3146 (April 1954).

7. M. J. LIGHTHILL, The response of laminar skin friction and heat transfer to fluctuations in the stream velocity, *Proc. R. Soc., Lond.* **224A**, No. 1156, 1–23 (June 1954).
8. F. K. MOORE, Unsteady laminar boundary layer flow, NACA TN 2471 (Sept. 1951).
9. S. I. CHENG, Some aspects of unsteady laminar boundary layer flows, *Q. Appl. Math.* **14**, 333–352 (January 1957).
10. S. I. CHENG and D. ELLIOTT, The unsteady laminar boundary layer on a flat plate, Princeton Univ. Aero. Eng. Dept. Rpt. No. 318 (Sept. 1955).
11. G. N. SARMA, Solutions of unsteady boundary layer problems, *Proc. Camb. Phil. Soc.* **60**, 137–158 (1964).
12. G. N. SARMA, A general theory of unsteady compressible boundary layers with and without suction or injection, *Proc. Camb. Phil. Soc.* **61**, 795–806 (1965).
13. G. N. SARMA, Unified theory for the solutions of the unsteady thermal boundary layer equations, *Proc. Camb. Phil. Soc.* **61**, 809–825 (1965).
14. G. N. SARMA, Solutions of unsteady compressible boundary layer equations, *Proc. Camb. Phil. Soc.* **62**, 511–518 (1966).
15. S. OSTRACH, Compressible laminar boundary layer and heat transfer for unsteady motions of a flat plate, NACA TN 3569 (Nov. 1955).
16. C. R. ILLINGWORTH, The effects of a sound wave on the compressible boundary layer on a flat plate, *J. Fluid Mech.* **3**, Part 5, 471–493 (Feb. 1958).
17. J. R. PESKIN and I. N. RABINOWITZ, Diftran I and II, Princeton Univ. Plasma Physics Lab. Tech. Memo. No. 154 (May 1962).
18. N. A. EVANS, Heat transfer through the unsteady laminary boundary layer on a semi-infinite flat plate, Towne School of Civil and Mechanical Engineering, University of Pennsylvania, Phila. (1972).

TRANSFERT THERMIQUE A TRAVERS LA COUCHE LIMITE LAMINAIRE INSTATIONNAIRE SUR UNE PLAQUE PLANE SEMI-INFINIE
lère partie: CONSIDERATIONS THEORIQUES

Résumé— On considère le problème d'une plaque semi-infinie se déplaçant à la vitesse $u_w(t)$ dans un fluide au repos et on obtient une solution du second ordre pour le transfert thermique avec enthalpie constante à la surface et pour une petite valeur de $\xi = x/\int_0^t u_e(t)\,dt$ (où t est le temps, $u_e(t) = -u_w(t)$ et x la distance au bord d'attaque). La solution, qui comprend l'effet de dissipation visqueuse, est valable pour $u_e(t)$ positif et infiniment différentiable pour tout t; elle est le prolongement du travail de Cheng et Elliot qui ont obtenu la solution du profil de vitesse et du frottement à partir de l'équation de quantité de mouvement pour fluide incompressible. Des résultats de second ordre sont obtenus par intégration numérique pour un nombre de Prandtl de 0, 72 et un accord satisfaisant existe avec les résultats d'autres chercheurs qui, utilisant différentes approches, ont obtenu seulement la solution de premier ordre ou une solution incomplète du second ordre.

WÄRMEÜBERTRAGUNG DURCH DIE INSTATIONÄRE LAMINARE GRENZSCHICHT AN EINER HALBUNENDLICHEN FLACHEN PLATTE
TEIL I: THEORETISCHE BETRACHTUNGEN

Zusammenfassung—Das Problem einer halbunendlichen Platte, die sich mit der Geschwindigkeit $u_w(t)$ in einer ruhenden Flüssigkeit bewegt, wird untersucht. Die dabei gewonnene Lösung für die Wärmeübertragung ist bei konstanter Oberflächenenthalpie korrekt bis zur zweiten Ordnung und gültig für kleine $\xi = x/\int_0^t u_e(t)\,dt$ (dabei ist t die Zeit, $u_e(t) = -u_w(t)$ und x die Entfernung von der Anströmkante). Die Lösung, die den Effekt der viskosen Dissipation berücksichtigt, ist gültig unter den Bedingungen $u_e(t) > 0$ und $u_e(t)$ unendlich oft differenzierbar für alle t.

Diese Arbeit ist eine Fortsetzung der Untersuchung von Cheng und Elliot, die die inkompressible Bewegungsgleichung für die Geschwindigkeitsprofile und die Wandreibung lösten Ergebnisse zweiter Ordnung erhält man durch numerische Integration für eine Prandtl-Zahl von 0,72; sie zeigen eine zufriedenstellende Übereinstimmung mit den Untersuchungen anderer Autoren, die unterschiedliche Näherungen verwendeten, entweder eine Lösung erster Ordnung oder eine unvollständige Lösung zweiter Ordnung.

ПЕРЕНОС ТЕПЛА ЧЕРЕЗ НЕСТАЦИОНАРНЫЙ ЛАМИНАРНЫЙ ПОГРАНИЧНЫЙ СЛОЙ НА ПОЛУБЕСКОНЕЧНОЙ ПЛОСКОЙ ПЛАСТИНЕ. 1. ТЕОРИЯ

Аннотация—Рассматривается задача о полубесконечной плоской пластине, движущейся в неподвижной жидкости со скоростью $u_w(t)$. С точностью до второго порядка получено решение для переноса тепла при постоянной энтальпии поверхности и небольших значений

$$\xi = x/\int_0^t u_e(t)\,\mathrm{d}t,$$

где t-время, $u_e(t) = -u_w(t)$, x-расстояние от передней кромки.

Решение, учитывающее вязкую диссипацию, справедливо при условии, что $u_e(t) > 0$, а $u_e(t)$ бесконечно дифференцируемо при всех t. Это решение является продолжением работы Ченга и Эллиота, которые решили уравнение импульса несжимаемой среды для профиля скоростей и поверхностного трения. Результаты с точностью до второго порядка получены численным интегрированием для числа Прандтля, равного 0,72. Показано удовлетворите льное совпадение с данными других исследователей, использующих иные методы.